Implicit differentiation:

Ex: Let z be defined implicitly by $x^3+y^3+z^3+6xyz=1$ $(*)$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Sol: apply $\frac{\partial}{\partial x}$ to $(*)$, keeping y constant:

$$3x^2+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{3x^2+6yz}{3z^2+6xy}=-\frac{x^2+2yz}{z^2+2xy}$$

apply $\frac{\partial}{\partial y}$ to $(*)$, keeping x constant:

$$3y^2+3z^2\frac{\partial z}{\partial y}+6xz+6xy\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{\partial z}{\partial y}=-\frac{y^2+2xz}{z^2+2xy}$$

If $y = f(x)$ is defined implicitly,
by $F(x, \underbrace{y}_{=f(x)}) = 0$

We can find $\frac{dy}{dx}$ via $0 = \frac{d}{dx} F(x,y) = \frac{\partial F}{\partial x} \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y} \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$$

Ex: y defined by $x^2 + y^3 = 1$ Find $y'(x)$.

Sol: $F(x,y) = x^2 + y^3 - 1 \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{3y^2}$

Ex: $z = z(x,y)$ defined by $x^2 + y^5 + z^3 = 0^{(*)}$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

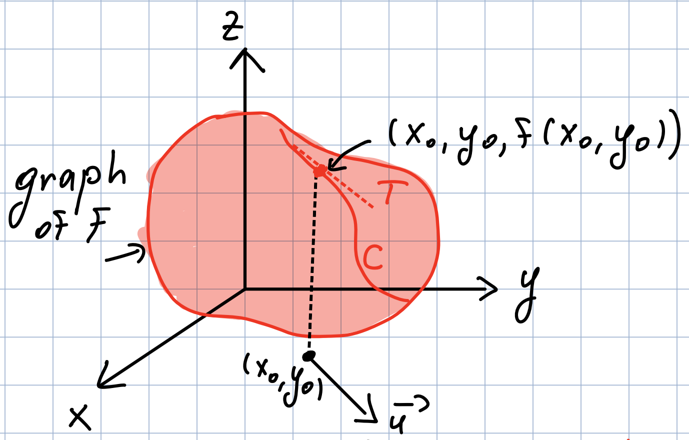
Sol: $\frac{\partial}{\partial x}(*)$: $2x + 0 + 3z^2 \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2x}{3z^2}$

$\frac{\partial}{\partial y}(*)$: $0 + 5y^4 + 3z^2 \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{5y^4}{3z^2}$

Directional derivatives, gradient

$f(x,y)$ $\quad$ $\vec{u} = \langle a, b \rangle$ $\quad$ $D_{\vec{u}} f(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$

unit vector



- directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$.
- rate of change of f in the direction of $\vec{u}$
= slope of tangent line T to the curve C.

Rmk: for $\vec{u} = \langle 1, 0 \rangle$, $D_{\vec{u}} f = f_x$

for $\vec{u} = \langle 0, 1 \rangle$, $D_{\vec{u}} f = f_y$

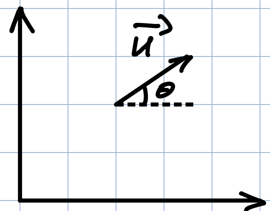
for $\vec{u} = \langle a, b \rangle$

$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b}$$

Idea: set $g(h) = f(x_0 + ha, y_0 + hb)$ - value of f on the line,

then $D_{\vec{u}} f(x_0, y_0) = g'(h)\Big|_{h=0} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dh} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dh}\Big|_{h=0}$

$$= f_x(x,y)\, a + f_y(x,y)\, b\Big|_{h=0} = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$



Unit vector: $\vec{u} = \langle \cos\theta, \sin\theta \rangle$

angle with x-axis

$$\Rightarrow D_{\vec{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

Ex: $f(x,y) = x^3 - 3xy + 4y^2$ $\quad$ $\vec{u}$ - unit vector given by $\theta = \pi/6$

Find $D_{\vec{u}} f(1,2)$.

Sol: $\vec{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle$

$$D_{\vec{u}} f(x,y) = f_x \cdot \frac{\sqrt{3}}{2} + f_y \cdot \frac{1}{2} = (3x^2 - 3y)\frac{\sqrt{3}}{2} + (8y - 3x)\frac{1}{2}$$

$$D_{\vec{u}} f(1,2) = (3-6)\frac{\sqrt{3}}{2} + (16-3)\frac{1}{2} = -\frac{3}{2}\sqrt{3} + \frac{13}{2}$$

Gradient: $D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b = \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$

$$= \boxed{\nabla f} \cdot \vec{u}$$

// $\underline{\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle} = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$ //

So, we have $\quad D_{\vec{u}} f(x,y) = \underbrace{\nabla f \cdot \vec{u}}_{\text{scalar proj. of } \nabla f \text{ onto } \vec{u}}$

Ex: $f(x,y) = x^2y^3 - 4y$. Find directional derivative at $(2,-1)$ in the direction of $\vec{v} = 2\vec{i} + 5\vec{j}$

Sol: 1) $\nabla f = \langle f_x, f_y \rangle = \langle 2xy^3, 3x^2y^2 - 4 \rangle$

$\nabla f(2,-1) = \langle -4, 8 \rangle$

2) $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{2\vec{i}+5\vec{j}}{\sqrt{4+25}} = \dfrac{\langle 2, 5\rangle}{\sqrt{29}}$ unit vector in the direction of $\vec{v}$

$\Rightarrow D_{\vec{u}} f = \nabla f \cdot \vec{u} = \langle -4, 8 \rangle \cdot \frac{1}{\sqrt{29}} \langle 2, 5 \rangle = \frac{-8}{\sqrt{29}} + \frac{40}{\sqrt{29}} = \frac{32}{\sqrt{29}}$

Ex: $f(\underbrace{x,y,z}_{\text{3 variables!}}) = x \sin yz$

a) Find $\nabla f = \langle f_x, f_y, f_z \rangle$

b) Find directional derivative of f at $(1,3,0)$ in the direction of $\vec{v} = \langle 1, 2, -1 \rangle$

Sol: (a) $\nabla f = \langle \sin yz,\ xz\cos yz,\ xy \cos yz \rangle$

(b) $\nabla f(1,3,0) = \langle 0, 0, 3 \rangle$

$\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{-1}{\sqrt{6}} \right\rangle$

$\Rightarrow D_{\vec{u}} f = \nabla f \cdot \vec{u} = -\frac{3}{\sqrt{6}}$

Q: In which direction does $f(x,y)$ (or $f(x,y,z)$) change fastest?
what is maximum rate of change?

A: Maximum value of directional derivative is $|\nabla f|$,
it occurs when $\vec{u}$ has same direction as $\nabla f(x,y)$

[Idea: $D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u} = |\nabla f| \underbrace{|\vec{u}|}_{=1} \underbrace{\cos\theta}_{\in[-1,1]}$ ← max at $\theta = 0$]

Ex: $f(x,y) = x^2 + y^2 \qquad (x_0, y_0) = (3, 4)$

Sol: $\nabla f(x,y) = \langle 2x, 2y \rangle \qquad \nabla f(3,4) = \langle 6, 8 \rangle$

$|\nabla f| = \sqrt{6^2 + 8^2} = 10$ - max rate of change

$\vec{u} = \dfrac{\langle 6, 8 \rangle}{10} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ - direction of fastest change

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Ex: $T(x,y,z) = \dfrac{40}{1 + x^2 + 2z^2}$ - temperature at (x,y,z)

a) In which direction the temp. increases fastest at the point $(1,0,1)$?

b) What is the max. rate of change?

$\underline{\text{Sol:}}$

$$\nabla T(x,y,z) = \langle T_x, T_y, T_z \rangle = \left\langle \frac{-40(2x)}{(1+x^2+2z^2)^2},\ 0,\ \frac{-40(4z)}{(1+x^2+2z^2)^2} \right\rangle$$

$$\nabla T(1,0,1) = \left\langle \frac{-40(2)}{(1+1^2+2(1)^2)^2},\ 0,\ \frac{-40(4)}{(1+1^2+2(1)^2)^2} \right\rangle = \langle -5, 0, -10 \rangle$$

$$|\nabla T(1,0,1)| = \sqrt{(-5)^2+0^2+(-10)^2} = \sqrt{125} = 5\sqrt{5}$$ - max rate of change

$$\vec{u} = \frac{\langle -5, 0, -10 \rangle}{5\sqrt{5}} = \left\langle -\frac{1}{\sqrt{5}},\ 0,\ -\frac{2}{\sqrt{5}} \right\rangle$$ - direction of fastest change